$$\mathcal{L}^c = \int_0^1 P_t(h)\cdot c_t(h)\,dh + P_t\left(C_t - \left[\int_0^1 c_t(h)^{\frac{\varepsilon-1}{\varepsilon}}\,dh\right]^{\frac{\varepsilon}{\varepsilon-1}}\right)$$

Derivative wrt $c_t(h)$

$$\frac{\partial \mathcal{L}^c}{\partial c_t(h)} = P_t(h) + P_t\left(\frac{\varepsilon}{\varepsilon-1}\right)\left[\int_0^1 c_t(h)^{\frac{\varepsilon-1}{\varepsilon}}\,dh\right]^{\frac{\varepsilon}{\varepsilon-1}-1}\cdot\left(\frac{\varepsilon-1}{\varepsilon}\right)\cdot c_t(h)^{\frac{\varepsilon-1}{\varepsilon}-1} \overset{!}{=} 0$$

$$C_t^{\frac{1}{\varepsilon}}$$

$$\Leftrightarrow \quad c_t(h) = \left(\frac{P_t(h)}{P_t}\right)^{-\varepsilon}\cdot C_t$$

Plugging into aggregation technology:

$$C_t^{\frac{\varepsilon-1}{\varepsilon}} = \int_0^1 c_t(h)^{\frac{\varepsilon-1}{\varepsilon}}\,dh$$

$$= \int_0^1 \left(\left(\frac{P_t(h)}{P_t}\right)^{-\varepsilon} C_t\right)^{\frac{\varepsilon-1}{\varepsilon}} dh$$

$$= C_t^{\frac{\varepsilon-1}{\varepsilon}}\cdot P_t^{\varepsilon-1}\int_0^1 P_t(h)^{1-\varepsilon}\,dh$$

$$\Leftrightarrow \quad P_t = \left[\int_0^1 P_t(h)^{1-\varepsilon}\,dh\right]^{\frac{1}{1-\varepsilon}}$$

$$\int_0^1 c_t(h)\, P_t(h)\, dh = \int_0^1 \left(\frac{P_t(h)}{P_t}\right)^{-\varepsilon} \cdot C_t \cdot P_t(h)\, dh$$

$$= P_t \cdot C_t \underbrace{\int_0^1 \left(\frac{P_t(h)}{P_t}\right)^{1-\varepsilon} dh}_{=1}$$

$$= P_t \cdot C_t$$

$$\mathcal{L}^{HH} = E_t \sum_{j=0}^{t} \beta^j \cdot U\left(c_{t+j}, n^s_{t+j}, z_{t+j}\right)$$

$$+ \beta^j \cdot \lambda_{t+j} \Big\{ \int_0^1 div_{t+j}(f)\, df$$

$$+ w_{t+j} \cdot n^s_{t+j}$$

$$+ \frac{B_{t-1+j}}{P_{t-1+j}} \cdot \frac{P_{t-1+j}}{P_{t+j}}$$

$$- Q_{t+j} \cdot \frac{B_{t+j}}{P_{t+j}}$$

$$- c_{t+j}$$

FOC wrt c_t:

$$\text{I} \quad \lambda_t = \frac{\partial u(c_t, n_t^s, z_t)}{\partial c_t} = z_t \cdot c_t^{-\sigma}$$

FOC wrt n_t^s:

$$w_t = -\frac{\partial u(c_t, n_t^s, z_t) / \partial n_t^s}{\lambda_t}$$

$$= n_t^{\varphi} \cdot c_t^{\sigma}$$

FOC wrt $b_t = \dfrac{B_t}{P_t}$

P_t / P_{t+1}

$$\lambda_t \cdot Q_t = \beta \cdot E_t\left[\lambda_{t+1} \cdot \Pi_{t+1}^{-1}\right]$$

Combine with I and $Q_t = \dfrac{1}{R_t}$

$$\frac{\partial u(c_t, n_t^s, z_t)}{\partial c_t} = \beta \cdot E_t\left[\frac{\partial u(c_{t+1}, n_{t+1}^s, z_{t+1})}{\partial c_{t+1}} \cdot \underbrace{\frac{R_t}{\Pi_{t+1}}}_{r_t}\right]$$

$$L^P = P_t \cdot y_t - \int_0^1 P_t(f) y_t(f) + \Lambda_t^P \left\{ \left[\int_0^1 y_t(f)^{\frac{\varepsilon-1}{\varepsilon}} df \right]^{\frac{\varepsilon}{\varepsilon-1}} - y_t \right\}$$

FOC wrt y_t:

$$\frac{\partial L^P}{\partial y_t} \overset{!}{=} 0 \iff P_t = \Lambda_t^P$$

FOC wrt $y_t(f)$:

$$\frac{\partial L^P}{\partial y_t(f)} = -P_t(f) + \Lambda_t^P \frac{\varepsilon}{\varepsilon-1} \left[\int_0^1 y_t(f)^{\frac{\varepsilon-1}{\varepsilon}} df \right]^{\frac{\varepsilon}{\varepsilon-1}-1} \left(\frac{\varepsilon-1}{\varepsilon} \right) y_t(f)^{\frac{\varepsilon-1}{\varepsilon}-1} = 0$$

Note: $\left[\int_0^1 y_t(f)^{\frac{\varepsilon-1}{\varepsilon}} df \right] = y_t^{\frac{\varepsilon-1}{\varepsilon}}$ and $\Lambda_t^P = P_t$

$$\Rightarrow P_t(f) = P_t \left[y_t^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon-\varepsilon+1}{\varepsilon-1}} \cdot y_t(f)^{\frac{\varepsilon-1-\varepsilon}{\varepsilon}} = P_t \left(\frac{y_t(f)}{y_t} \right)^{-\frac{1}{\varepsilon}}$$

$$\iff y_t(f) = \left(\frac{P_t(f)}{P_t} \right)^{-\varepsilon} y_t$$

Aggregate price index implicitly defined:

$$y_t = \left[\int_0^1 \left(\left(\frac{P_t(f)}{P_t} \right)^{-\varepsilon} y_t \right)^{\frac{\varepsilon-1}{\varepsilon}} df \right]^{\frac{\varepsilon}{\varepsilon-1}}$$

$$\iff P_t = \left[\int_0^1 P_t(f)^{1-\varepsilon} df \right]^{\frac{1}{1-\varepsilon}}$$

$$\mathcal{L}^f = E_t \sum_{j=0}^{\infty} \Lambda_{t,t+j} \cdot P_{t+j} \left[\frac{P_{t+j}(f)}{P_{t+j}} \cdot y_{t+j}(f) - w_{t+j} \cdot n^d_{t+j}(f) + mc_{t+j}(f) \left(a_{t+j} \cdot n^d_{t+j}(f) - y_{t+j}(f) \right) \right]$$

$$\frac{\partial \mathcal{L}^f}{\partial n^d_t(f)} = 0$$

$$\Leftrightarrow \quad w_t = mc_t(f) \cdot a_t = mc_t(f) \cdot \frac{y_t(f)}{n^d_t(f)}$$

$$mc_t(f) = \frac{w_t}{a_t} \Rightarrow \text{independent of } f$$

$$mc_t = \int_0^1 mc_t(f)\, df = \frac{w_t}{a_t}$$

$$\mathcal{L}^f = E_t \sum_{j=0}^{\infty} \theta^j \Lambda_{t,t+j} \cdot P_{t+j} \left[\left(\frac{\tilde{P}_t(f)}{P_{t+j}} \right)^{1-\varepsilon} \cdot y_{t+j} - w_{t+j} \cdot n^d_{t+j}(f) + mc_{t+j} \left(a_{t+j} \cdot n^d_{t+j}(f) - \left(\frac{\tilde{P}_t(f)}{P_{t+j}} \right)^{-\varepsilon} \cdot y_{t+j} \right) \right]$$

$$= E_t \sum_{j=0}^{\infty} \theta^j \Lambda_{t,t+j} \cdot P^{\varepsilon}_{t+j} \cdot y_{t+j} \left[\tilde{P}_t(f)^{1-\varepsilon} + P_{t+j} \cdot mc_{t+j} \tilde{P}_t(f)^{-\varepsilon} \right]$$

$$+ \ldots$$

FOC wrt $\tilde{P}_t(f)$:

$$0 = E_t \sum_{j=0}^{\infty} \theta^j \Lambda_{t,t+j} P_{t+j}^{\varepsilon} y_{t+j} \left[(1-\varepsilon) \tilde{P}_t(f)^{-\varepsilon} + \varepsilon P_{t+j} \cdot mc_{t+j} \cdot \tilde{P}_t(f)^{-\varepsilon-1} \right]$$

Multiply by $\tilde{P}_t(f)^{\varepsilon+1}$

$$0 = E_t \sum_{j=0}^{\infty} \theta^j \Lambda_{t,t+j} P_t^{\varepsilon} \cdot y_{t+j} \left[(1-\varepsilon) \tilde{P}_t(f) + \varepsilon \cdot P_{t+j} \cdot mc_{t+j} \right]$$

$$\Leftrightarrow \tilde{P}_t(f) \cdot E_t \sum_{j=0}^{\infty} \theta^j \Lambda_{t,t+j} P_t^{\varepsilon} \cdot y_{t+j}$$

$$= \frac{\varepsilon}{\varepsilon - 1} E_t \sum_{j=0}^{\infty} \theta^j \Lambda_{t,t+j} \cdot P_{t+j}^{\varepsilon+1} \cdot y_{t+j} \cdot mc_{t+j}$$

Dividing by $P_t^{\varepsilon+1}$:

$$\underbrace{\frac{\tilde{P}_t(f)}{P_t}}_{\tilde{p}_t} \underbrace{E_t \sum_{j=0}^{\infty} \theta^j \Lambda_{t,t+j} \left(\frac{P_{t+j}}{P_t}\right)^{\varepsilon} y_{t+j}}_{S_{1,t}} = \frac{\varepsilon}{\varepsilon-1} \underbrace{E_t \sum_{j=0}^{\infty} \theta^j \Lambda_{t,t+j} \cdot \left(\frac{P_{t+j}}{P_t}\right)^{\varepsilon+1} \cdot y_{t+j} \cdot mc_{t+j}}_{S_{2,t}}$$

→ All firms set the same price

$$\tilde{p}_t \cdot S_{1,t} = \frac{\varepsilon}{\varepsilon-1} \cdot S_{2,t}$$

Recursive $S_{1,t}$:

$$S_{1,t} = E_t \sum_{j=0}^{\infty} \theta^{j} \Lambda_{t,t+j} \left(\frac{P_{t+j}}{P_t}\right)^{\varepsilon} \cdot y_{t+j}$$

$$= y_t + E_t \sum_{j=1}^{\infty} \theta^{j} \Lambda_{t,t+j} \left(\frac{P_{t+j}}{P_t}\right)^{\varepsilon} \cdot y_{t+j}$$

$$= y_t + E_t \sum_{j=0}^{\infty} \theta^{j+1} \Lambda_{t,t+j+1} \left(\frac{P_{t+j+1}}{P_t}\right)^{\varepsilon} \cdot y_{t+j+1}$$

$$= y_t + E_t \sum_{j=0}^{\infty} \theta^{j+1} \Lambda_{t,t+j+1} \left(\frac{P_{t+j+1}}{P_{t+1}} \frac{P_{t+1}}{P_t}\right)^{\varepsilon} \cdot y_{t+j+1}$$

$$= y_t + E_t \sum_{j=0}^{\infty} \theta^{j+1} \beta \cdot \frac{\lambda_{t+1}}{\lambda_t} \Pi_{t+1}^{-1} \Lambda_{t,t+j+1} \left(\frac{P_{t+j+1}}{P_{t+1}} \cdot \Pi_t\right)^{\varepsilon} \cdot y_{t+j+1}$$

$$= y_t + E_t \theta \beta \frac{\lambda_{t+1}}{\lambda_t} \Pi_{t+1}^{\varepsilon-1} \underbrace{E_t \sum_{j=0}^{\infty} \theta^{j} \Lambda_{t+1,t+1+j} \left(\frac{P_{t+j+1}}{P_{t+1}}\right)^{\varepsilon} \cdot y_{t+j+1}}_{S_{1,t+1}}$$

$$\Leftrightarrow S_{1,t} = y_t + E_t \theta \beta \frac{\lambda_{t+1}}{\lambda_t} \cdot \Pi_{t+1}^{\varepsilon-1} \cdot S_{1,t+1}$$

$$S_{2,t} = E_t \sum_{j=0}^{\infty} \theta^{j} \Lambda_{t,t+j} \left(\frac{P_{t+j}}{P_t}\right)^{\varepsilon+1} \cdot y_{t+j} \cdot mc_{t+j}$$

$$= y_t \cdot mc_t + E_t \sum_{j=1}^{\infty} \theta^{j} \Lambda_{t,t+j} \left(\frac{P_{t+j}}{P_t}\right)^{\varepsilon+1} \cdot y_{t+j} \cdot mc_{t+j}$$

$$= y_t \cdot mc_t + E_t \sum_{j=0}^{\infty} \theta^{j+1} \Lambda_{t,t+j+1} \left(\frac{P_{t+j+1}}{P_t}\right)^{\varepsilon+1} \cdot y_{t+j+1} \cdot mc_{t+j+1}$$

$$= y_t \cdot mc_t + E_t \sum_{j=0}^{\infty} \theta^{j+1} \Lambda_{t,t+j+1} \left(\frac{P_{t+j+1}}{P_{t+1}} \frac{P_{t+1}}{P_t}\right)^{\varepsilon+1} \cdot y_{t+j+1} \cdot mc_{t+j+1}$$

$$= y_t \cdot mc_t + E_t \sum_{j=0}^{\infty} \theta^{j+1} \beta \frac{\lambda_{t+1}}{\lambda_t} \cdot \Pi_{t+1}^{-1} \Lambda_{t+1,t+1+j} \left(\frac{P_{t+j+1}}{P_{t+1}} \frac{P_{t+1}}{P_t}\right)^{\varepsilon+1} \cdot y_{t+j+1} \cdot mc_{t+j+1}$$

$$= y_t \cdot mc_t + \theta \beta \underbrace{E_t \frac{\lambda_{t+1}}{\lambda_t} \Pi_{t+1}^{\varepsilon} E_t \sum_{j=0}^{\infty} \theta^{j} \Lambda_{t+1,t+1+j} \left(\frac{P_{t+j+1}}{P_{t+1}}\right)^{\varepsilon+1} \cdot y_{t+j+1} \cdot mc_{t+j+1}}_{S_{2,t+1}}$$

$$\Longleftrightarrow S_{2,t} = y_t \cdot mc_t + \theta \beta \cdot S_{2,t+1}$$

$$1 = \int_0^1 \left(\frac{P_t(f)}{P_t}\right)^{1-\varepsilon} df$$

$$1 = \int_{\text{optim}} \left(\frac{\tilde{P}_t}{P_t}\right)^{1-\varepsilon} df + \int_{\text{non-optim}} \left(\frac{P_t(f)}{P_t}\right)^{1-\varepsilon} df$$

$$= (1-\theta)\left(\frac{\tilde{P}_t}{P_t}\right)^{1-\varepsilon} + \theta \cdot \int_0^1 \left(\frac{P_{t-1}(f)}{P_t} \cdot \frac{P_{t-1}}{P_{t-1}}\right)^{1-\varepsilon} df$$

$$= (1-\theta)\left(\tilde{p}_t\right)^{1-\varepsilon} + \theta \cdot \left(\frac{P_{t-1}}{P_t}\right)^{1-\varepsilon} \underbrace{\int_0^1 \left(\frac{P_{t-1}(f)}{P_{t-1}}\right)^{1-\varepsilon} df}_{=1}$$

$\Longleftrightarrow$

$$1 = (1-\theta)(\tilde{p}_t)^{1-\varepsilon} + \theta \cdot \Pi_t^{\varepsilon-1}$$

Market clearing

$B_t = 0 \implies$ Always

$$n_t^s = \int_0^1 n_t^d(f)\, df = n_t$$

$$y_t(f) = \int_0^1 \left(\frac{p_t(f)}{p_t}\right)^{-\varepsilon} y_t \, df$$

$$\int_0^1 y_t(f) \cdot p_t(f) \, df = \int_0^1 \left(\frac{p_t(f)}{p_t}\right)^{-\varepsilon} \cdot y_t \cdot p_t(f) \, df$$

$$= p_t \cdot y_t \underbrace{\int_0^1 \left(\frac{p_t(f)}{p_t}\right)^{1-\varepsilon} df}_{=1}$$

$$= p_t \cdot y_t$$

$$div_t = \int_0^1 div_t(f) \, df = \int_0^1 \frac{p_t(f)}{p_t} \cdot y_t(f) \, df - \int_0^1 w_t n_t^d(f) \, df$$

$$= y_t - w_t \cdot n_t$$

Revisit budget constraint:

$$\int_0^1 \frac{p_t(h)}{p_t} c_t(h) \cdot dh + Q_t \cdot b_t = b_{t-1} \cdot \Pi_t^{-1} + w_t \cdot n_t^s + \int_0^1 div_t(f) df$$

$$\Leftrightarrow \frac{P_t \cdot c_t}{P_t} = w_t \cdot n_t + y_t - w_t \cdot n_t$$

$$\Leftrightarrow c_t = y_t$$

Aggregate supply

$$y_t^{sum} = \int_0^1 y_t(f)\, df$$

$$= \int_0^1 a_t \cdot n_t^d(f)\, df = a_t \cdot n_t$$

But also: $y_t(f) = \left(\frac{P_t(f)}{P_t}\right)^{-\varepsilon} y_t \, df$

$$y_t^{sum} = \int_0^1 y_t(f)\, df$$

$$= y_t \underbrace{\int_0^1 \left(\frac{P_t(f)}{P_t}\right)^{-\varepsilon} df}_{p_t^*}$$

Equating both y_t^{sum}:

$$p_t^* \cdot y_t = a_t \cdot n_t \qquad p_t^* \leq 1$$

Calvo mechanism:

$$P_t^* = \int_0^1 \left(\frac{P_t(f)}{P_t}\right)^{-\varepsilon} df$$

$$= \int_{\text{OPTIM}} \left(\frac{P_t(f)}{P_t}\right)^{-\varepsilon} df + \int_{\text{nonOPTIM}} \left(\frac{P_t(f)}{P_t}\right)^{-\varepsilon} df$$

$$= (1-\theta)\cdot\left(\hat{P}_t\right)^{-\varepsilon} + \theta \int_0^1 \left(\frac{P_{t-1}(f)}{P_t}\cdot\frac{P_{t-1}}{P_{t-1}}\right)^{-\varepsilon} df$$

$$= (1-\theta)\left(\hat{P}_t\right)^{-\varepsilon} + \theta\cdot\left(\frac{P_{t-1}}{P_t}\right)^{-\varepsilon}\cdot\underbrace{\int_0^1 \left(\frac{P_{t-1}(f)}{P_{t-1}}\right)^{-\varepsilon} df}_{P^*_{t-1}}$$

$$\iff P_t^* = (1-\theta)\left(\hat{P}_t\right)^{-\varepsilon} + \theta\,\Pi^{\varepsilon}\cdot P^*_{t-1}$$